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STEADY-STATE ENTANGLEMENT CLASS IN FINITE-DIMENSIONALQUANTUM SYSTEMS CANNOT BE DETERMINED BY FULL NON-ZERO LIOUVILLIAN SPECTRA

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Abstract

We construct two finite-dimensional, exactly solvable two-qubit Lindbladian systems possessing strict two-dimensional invariant manifolds. For both models, we derive closed-form steady states, complete sets of non-zero Liouvillian eigenvalues with multiplicities, and analytic concurrence expressions. We show that, under parameter matching, the two generators possess identical full non-zero Liouvillian spectra, including both relaxation rates (real parts) and coherent oscillation frequencies (imaginary parts), while converging to steady states with locally unitarily inequivalent entanglement structure. This establishes constructively that full non-zero Liouvillian spectral data are insufficient to determine steady-state entanglement class, because the non-zero spectrum governs contraction transverse to the invariant manifold rather than its internal geometric resource structure.

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1 Introduction

Quantum systems with environmental coupling evolve according to completely positive trace-preserving semigroups generated by Lindblad operators. The spectrum of the Liouvillian generator governs transient dynamics toward steady states: real parts of eigenvalues set relaxation rates, with the smallest non-zero real part defining the Liouvillian gap, while imaginary parts encode coherent oscillatory structure.

Spectral properties of the Liouvillian are widely used as dynamical diagnostics in dissipative physics, including relaxation-time analysis, dissipative criticality, and resource persistence. It is therefore natural to ask whether the full non-zero Liouvillian spectrum, encompassing both relaxation and oscillatory timescales, encodes sufficient information to determine the geometric and entanglement structure of the steady state.

We provide a constructive negative answer. Two analytically solvable two-qubit Lindbladians are presented whose full non-zero eigenvalue sets coincide exactly under parameter matching, yet whose steady states possess inequivalent entanglement properties.

2 Structural Framework

Let a finite-dimensional Lindbladian admit a decomposition of operator space

$$\mathcal{B}(\mathcal{H}) = \mathcal{M} \oplus \mathcal{M}^{\perp}, \tag{1}$$

such that:

1. The invariant manifold $\mathcal{M}$ contains the unique steady state.
2. Dissipators produce strict contraction on $\mathcal{M}^{\perp}$.
3. The Hamiltonian acts only within $\mathcal{M}$.

In Liouville representation, the generator admits block form

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_{\mathcal{M}} & 0 \\ * & \mathcal{L}_{\perp} \end{pmatrix}. \tag{2}$$

The non-zero eigenvalues are determined entirely by contraction on transverse sectors, while steady-state geometric resources are fixed by the kernel structure on the invariant manifold.

3 Model I: Bell-Invariant Manifold

Bell basis:

$$|\psi^{\pm}\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}, \tag{3}$$

$$|\phi^{\pm}\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}. \tag{4}$$

Invariant manifold:

$$\mathcal{M}_1 = \mathrm{span}\{|\psi^-\rangle, |\phi^-\rangle\}. \tag{5}$$

Dissipators:

$$L_1 = \sqrt{\gamma}\,|\psi^-\rangle\langle\psi^+|, \tag{1}$$

$$L_2 = \sqrt{\gamma}\,|\psi^-\rangle\langle 00|, \tag{2}$$

$$L_3 = \sqrt{\gamma}\,|\psi^-\rangle\langle 11|. \tag{3}$$

Hamiltonian:

$$H_1 = \epsilon\big(\sigma_x^{(1)} - \sigma_x^{(2)}\big). \tag{6}$$

Assume $\gamma > 0$ and, without loss of generality, $\epsilon \geq 0$ (a sign flip of ϵ changes only an internal phase convention). Define $r = \epsilon/\gamma \geq 0$. The steady state in $\mathcal{M}_1$ is

$$\rho_1 = \begin{pmatrix} a(r) & iy(r) \\ -iy(r) & 1 - a(r) \end{pmatrix}, \tag{7}$$

where

$$a(r) = \frac{16r^2 + 1}{32r^2 + 1}, \qquad y(r) = -\frac{4r}{32r^2 + 1}. \tag{8}$$

Concurrence:

$$C_1(r) = \frac{\sqrt{1 + 64r^2}}{32r^2 + 1}. \tag{9}$$

Full non-zero Liouvillian spectrum (with multiplicities):

$$\lambda_a = -\gamma \quad \text{(mult. 4)}, \tag{4}$$

$$\lambda_b = -\frac{\gamma}{2} \quad \text{(mult. 1)}, \tag{5}$$

$$\lambda_{c,\pm} = -\frac{3\gamma}{4} \pm \frac{\gamma}{4}\sqrt{1 - 64r^2} \quad \text{(each mult. 4)}, \tag{6}$$

$$\lambda_{d,\pm} = -\frac{3\gamma}{4} \pm \frac{\gamma}{4}\sqrt{1 - 256r^2} \quad \text{(each mult. 1)}, \tag{7}$$

plus a single zero eigenvalue.

For $r > 1/8$, the eigenvalues $\lambda_{c,\pm}$ acquire non-zero imaginary parts and form complex conjugate pairs. Similarly, for $r > 1/16$, the eigenvalues $\lambda_{d,\pm}$ become complex conjugate pairs.

4 Model II: Spectrum-Matched Computational Manifold

Define a fixed nonlocal unitary U by

$$U|\psi^-\rangle = |00\rangle, \tag{8}$$
$$U|\phi^-\rangle = |11\rangle, \tag{9}$$
$$U|\psi^+\rangle = |01\rangle, \tag{10}$$
$$U|\phi^+\rangle = |10\rangle. \tag{11}$$

Construct Model II by conjugation:

$$H_2 = UH_1U^\dagger, \qquad L_k^{(2)} = UL_kU^\dagger. \tag{10}$$

An equivalent explicit representation is:

$$H_2 = -2\epsilon\big(|00\rangle\langle 11| + |11\rangle\langle 00|\big), \tag{12}$$
$$J_1 = \sqrt{\gamma}\,|00\rangle\langle 01|, \tag{13}$$
$$J_2 = \sqrt{\gamma}\,|00\rangle\langle 10|, \tag{14}$$
$$J_3 = \sqrt{\gamma}\,|00\rangle\langle 11|. \tag{15}$$

Because the superoperators satisfy

$$\mathcal{L}_2 = \mathcal{U}\,\mathcal{L}_1\,\mathcal{U}^{-1}, \tag{11}$$

Models I and II are exactly isospectral on full Liouville space, including all non-zero eigenvalues and multiplicities.

The steady state is $\rho_2 = U\rho_1U^\dagger$, with concurrence

$$C_2(r) = \frac{8r}{32r^2 + 1}. \tag{12}$$

5 Spectral Equivalence and Entanglement Inequivalence

Proposition 5.1. *For every fixed $r \geq 0$, Models I and II have identical full non-zero Liouvillian spectra (including multiplicities, real parts, and imaginary parts), but their steady states are not locally unitarily equivalent. In particular,*

$$C_1(r) - C_2(r) = \frac{\sqrt{1+64r^2} - 8r}{32r^2+1} > 0. \tag{13}$$

Proof. Similarity $\mathcal{L}_2 = \mathcal{U}\mathcal{L}_1\mathcal{U}^{-1}$ implies equality of full spectra, hence equality of all non-zero spectral data and multiplicities. On the other hand, concurrence is invariant under local unitaries on two qubits. Since $r \geq 0$ we have $\sqrt{1+64r^2} > \sqrt{64r^2} = 8r$, so the numerator is strictly positive and the denominator is positive. Therefore $C_1(r) > C_2(r)$ for all $r \geq 0$, proving local-unitary inequivalence of the steady states despite complete non-zero spectral coincidence.

At $r = 1/4$ this gives

$$C_1(1/4) = \frac{\sqrt{5}}{3}, \qquad C_2(1/4) = \frac{2}{3}, \tag{14}$$

consistent with the strict inequality above. □

6 Conclusion

We have provided an explicit finite-dimensional counterexample showing that full non-zero Liouvillian spectra, including real and imaginary parts with multiplicities, are insufficient to classify steady-state entanglement class. The non-zero spectrum encodes transverse contraction structure; steady-state entanglement is a kernel-manifold geometric property.

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